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Perturbed-Law based sensitivity Indices for sensitivity analysis in structural reliability

8th International Conference

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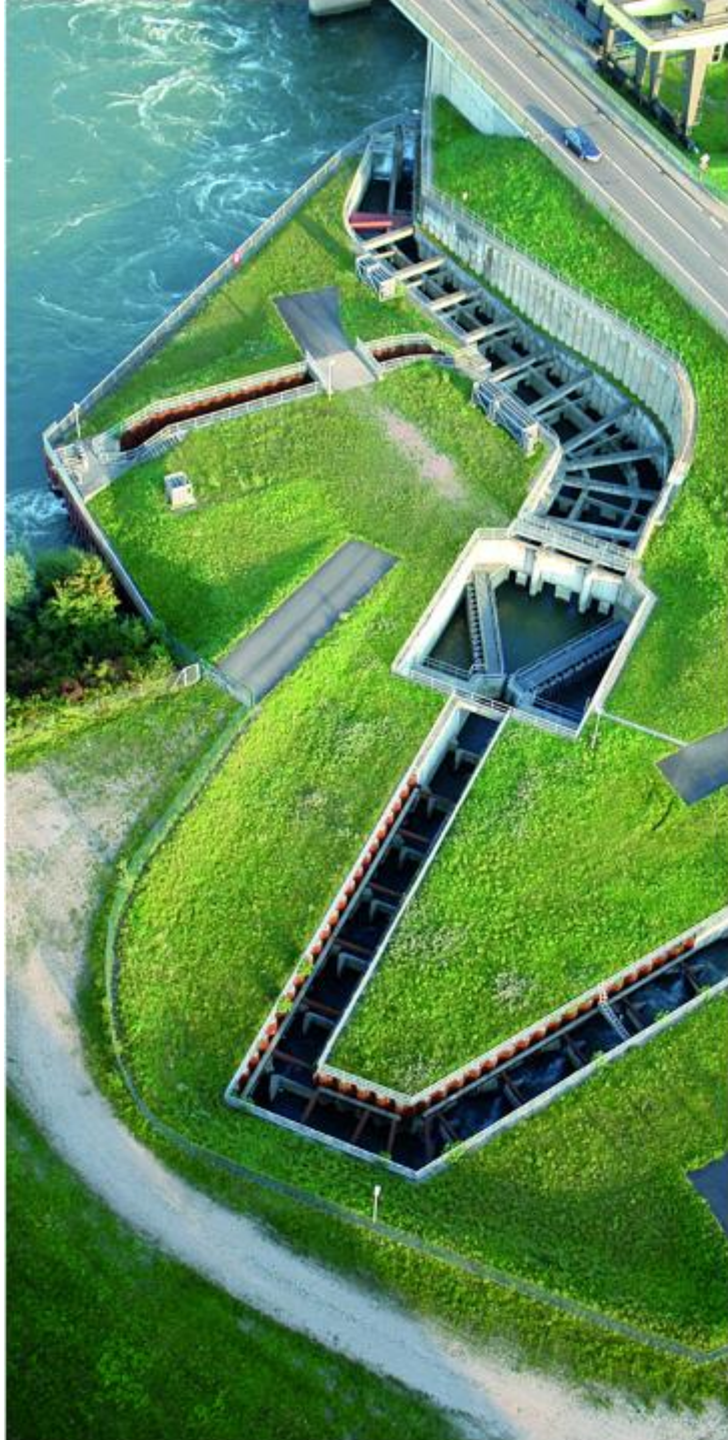
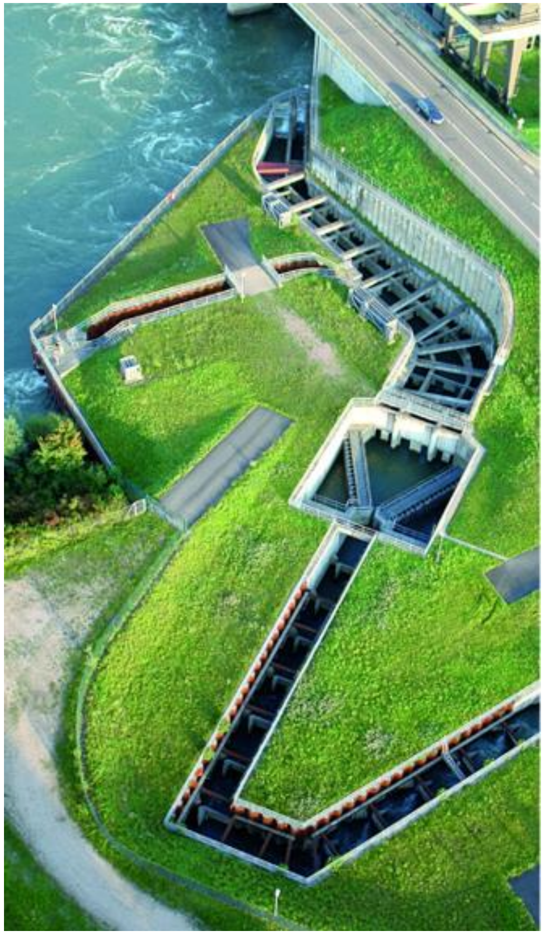
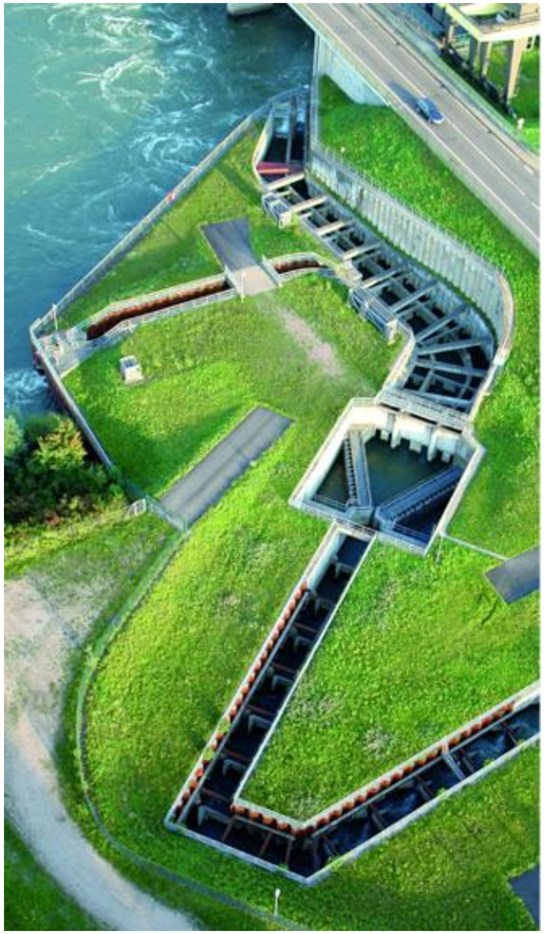






1. Context : sensitivity analysis on a failure probability

CONTEXT OF STRUCTURAL RELIABILITY

We want to study a computer code G which :

- **is deterministic**
- **is a « costly » numerical model (CPU time, memory,…)**
- **has d input variables**
- **allows calculating the value $G(\boldsymbol{X})$ of the reliability criterion for a given set of input values $\boldsymbol{X} = (X_1, \dots, X_d)$**

The input variables are uncertain, hence we denote

- $\mathbb{X} \subset \mathbb{R}^d$ **the domain of variation of the random vector $\boldsymbol{X}$**
- $f = \prod_{i=1}^{d} f_i$ **the probability density function of $\boldsymbol{X}$**
  - each f_i is the density of X_i, the i-th marginal of $\boldsymbol{X}$
  - the uncertain input variables $X_1, \dots X_d$ are considered independent

CONTEXT OF STRUCTURAL RELIABILITY

In particular, we get interested in the following event

$$\{G(\boldsymbol{X}) < 0\}$$

► **binary quantity of interest**

We aim at calculating a « failure probability » :

$$P_f = \int_{\mathbb{X}} \mathbb{I}_{\{G(\boldsymbol{x})<0\}} f(\boldsymbol{x}) d\boldsymbol{x}$$

Problem : how to study the influence of the input variables on P_f ?

Usually, P_f is low ($< 10^{-3}$)

CONTEXT OF STRUCTURAL RELIABILITY

$$P_f = \int_{\mathbb{X}} \mathbb{I}_{\{G(\boldsymbol{x})<0\}} f(\boldsymbol{x}) d\boldsymbol{x}$$

Problem : how to study the influence of the input variables on P_f ?

- Many techniques aim at detecting most influencial inputs on a quantity of interest on $Y = G(\boldsymbol{X})$ (usually assumed continuous with respect to each X_i)
- The alternative approach we're choosing here consists in quantifying the impact of the lack of knowledge about the law of $\boldsymbol{X}$

A FIRST IDEA : SOBOL' INDICES FOR $x \mapsto \mathbb{I}_{\{G(x)<0\}}$

► **good method to distinguish influential and non-influential variables**

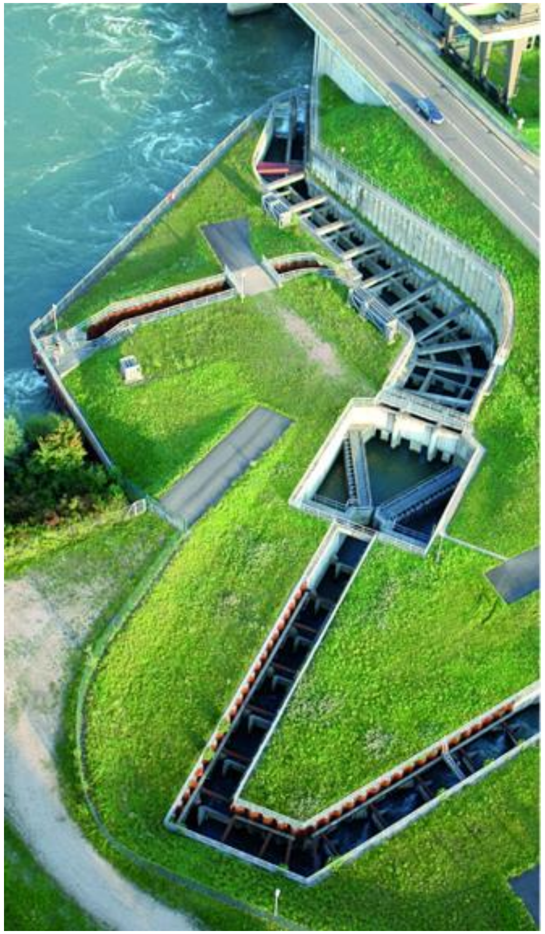
In general :

- Weak first order indices
- Strong total indices

assesses that no variable is influential on its own, and that most variables contribute to the failure probability when interacting with the others

► **in most structural reliability cases, this is an already known information : the equipment fails when all variable take extreme values at the same time**

+ this information is obtained at a high computational cost



1. Context : sensitivity analysis on a failure probability
2. Principle of PLI indices

PLI INDICES : THE PRINCIPLE

For a particular random input X_i which's density is f_i, we introduce a δ perturbation, leading to a perturbed density $f_{i\delta}$

The failure probability becomes :

$$P_{i\delta} = \int_{\mathbb{X}} \mathbb{I}_{\{G(\boldsymbol{x})<0\}} f(\boldsymbol{x}) \frac{f_{i\delta}(x_i)}{f_i(x_i)} d\boldsymbol{x}$$

The corresponding PLI indice is defined by :

$$S_{i\delta} = \left(\frac{P_{i\delta}}{P_f} - 1\right) \mathbb{I}_{\{P_{i\delta} > P_f\}} + \left(1 - \frac{P_f}{P_{i\delta}}\right) \mathbb{I}_{\{P_{i\delta} < P_f\}}$$

Introduced in *Lemaître et al. (2015)*

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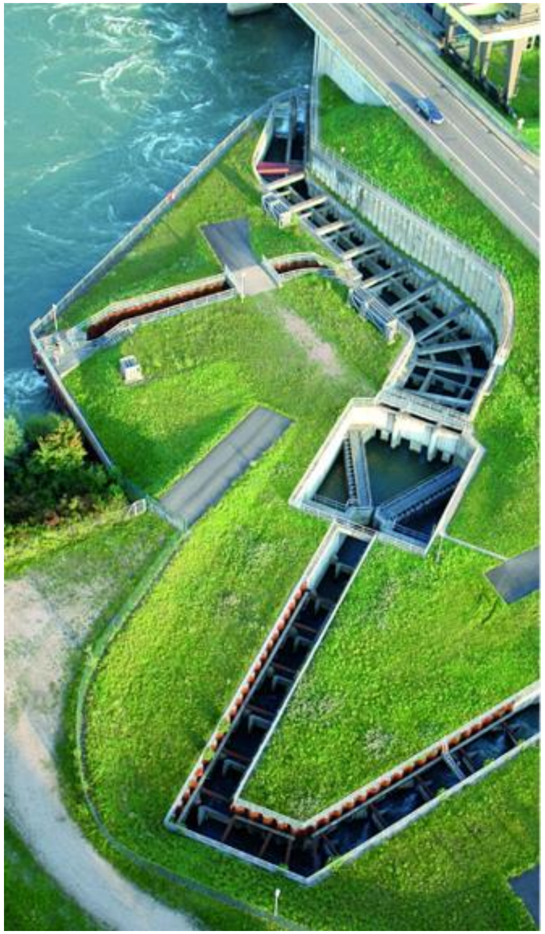
- $S_{i\delta} = 0$ when $P_{i\delta} = P_f$ e.g. when berturbating f_i has no impact on the failure probability
- The sign of $S_{i\delta}$ indicates how the perturbation modifies the failure probability (symetrical behaviour for increase or decrease of the output probability)



1. Context : sensitivity analysis on a failure probability
2. Principle of PLI indices
3. **Density perturbation**

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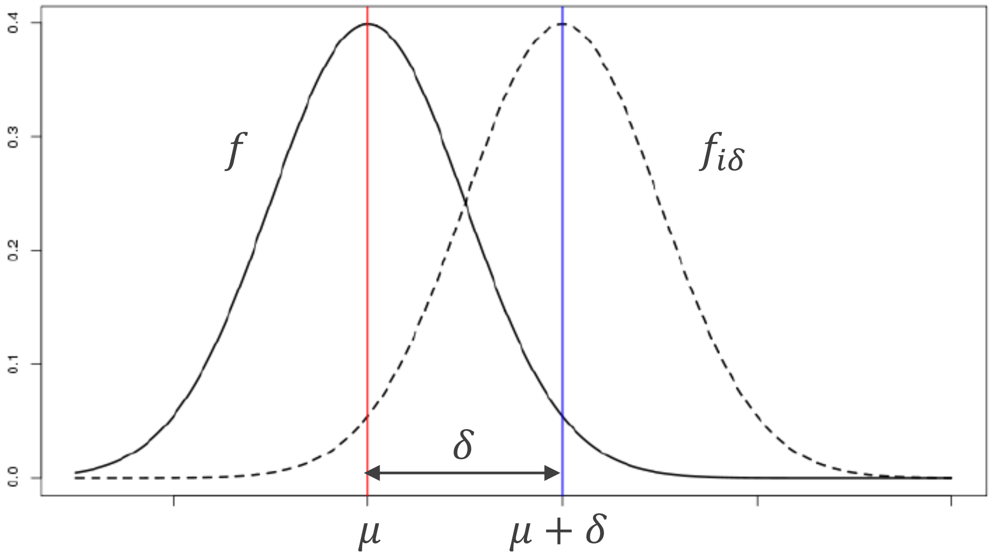
HOW TO DEFINE A DENSITY PERTURBATION ?

- **Let's assume that the X_i input variable has a normal distribution $X_i \sim \mathcal{N}(\mu, \sigma^2)$**
- **What if the mean of X_i was not μ but $\mu + \delta$?**



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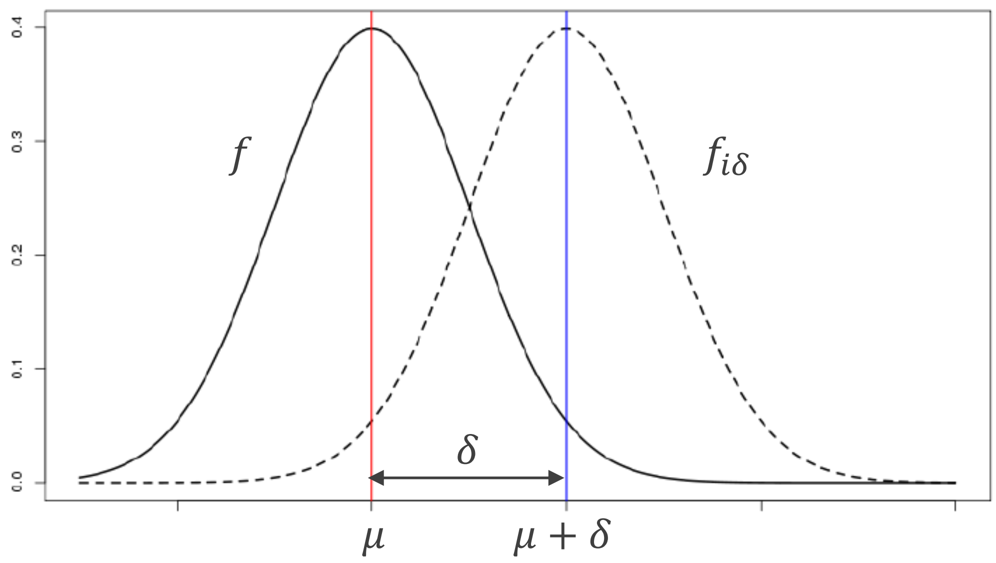
- **Let's assume that the X_i input variable has a normal distribution $X_i \sim \mathcal{N}(\mu, \sigma^2)$**
- **How to define $f_{i\delta}$ with the constraint $\int_{\mathbb{X}_i} x_i f_{i\delta}(x_i)dx_i = \mu + \delta$?**



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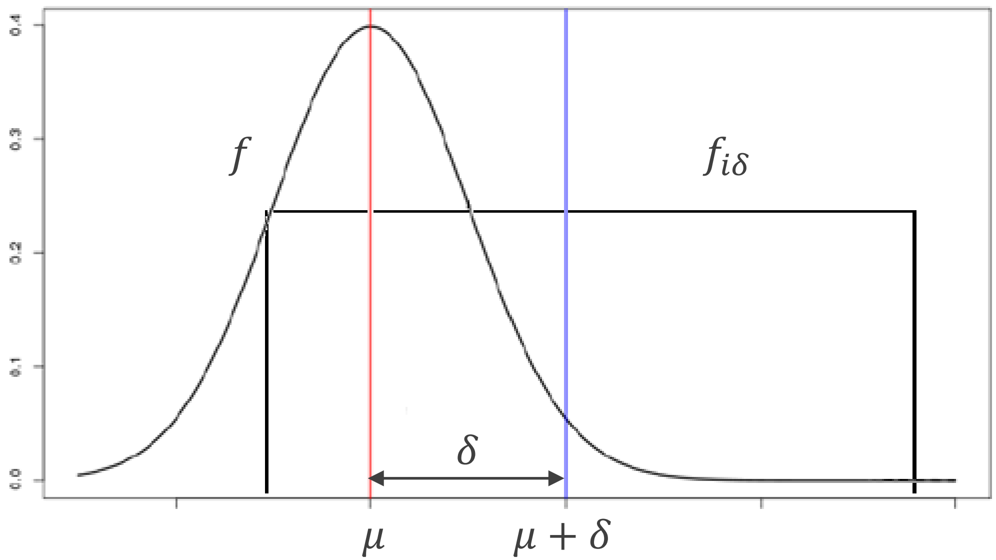

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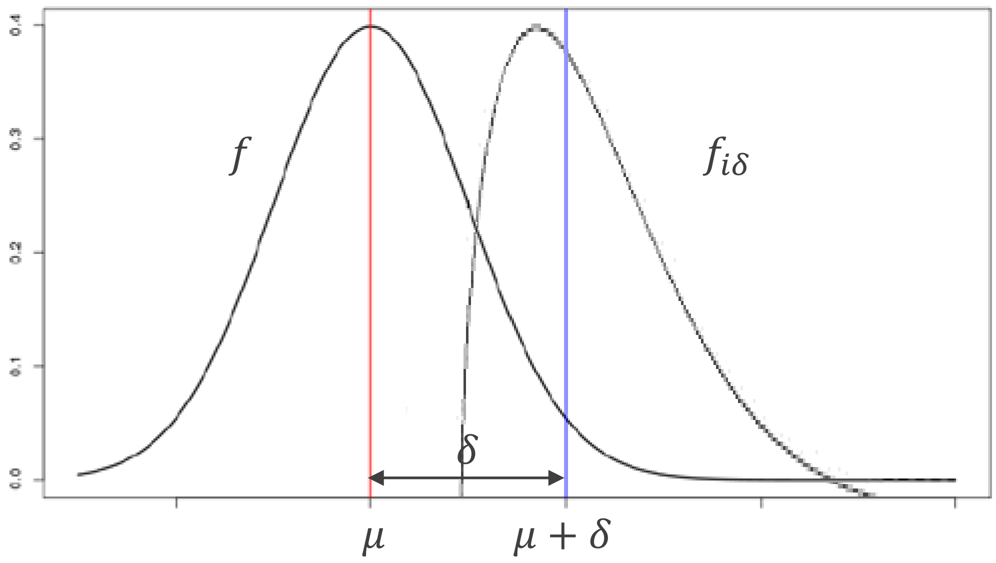

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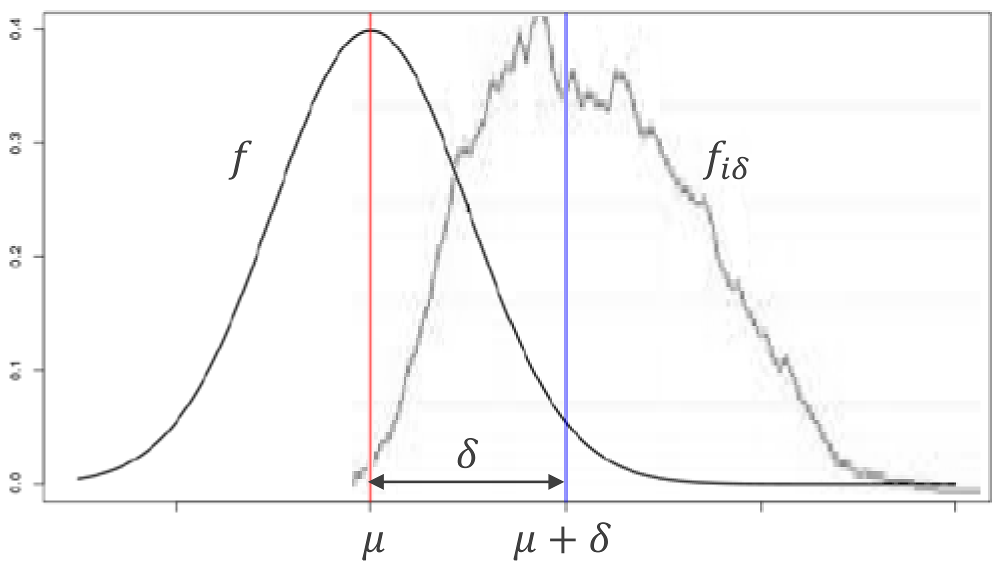

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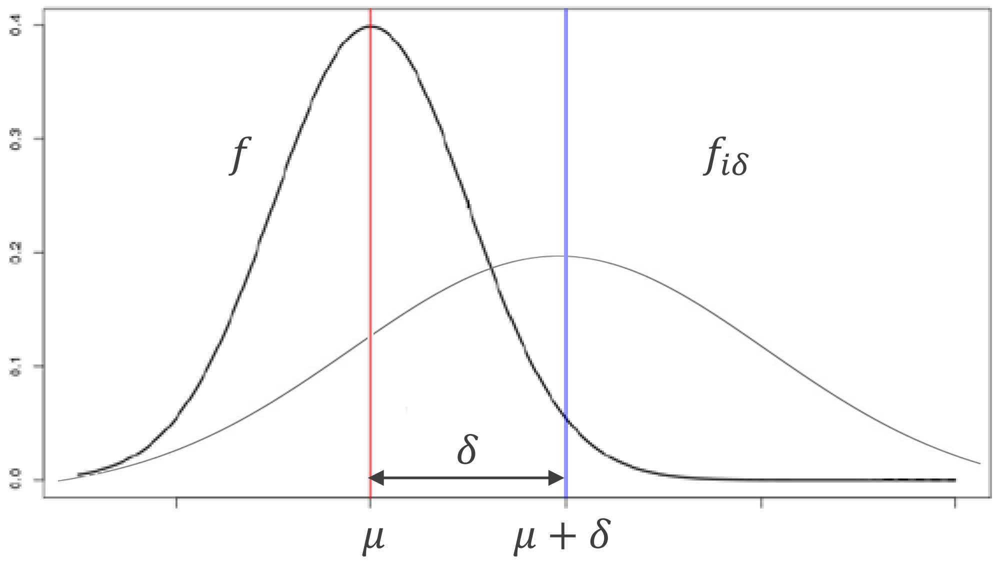


HOW TO DEFINE A DENSITY PERTURBATION ?

- We suggest to define the perturbed density $f_{i\delta}$ as **the closest one from the initial f_i** in the sense of the entropy, **under the constraint of perturbation**

- Recall that :

$$KL(\pi_1, \pi_2) = \int_{-\infty}^{+\infty} \pi_1(x) \log\left(\frac{\pi_1(x)}{\pi_2(x)}\right) dx$$

- So we can give a general formal definition for $f_{i\delta}$ the following way :

$$f_{i\delta} = \underset{\substack{\pi \\ s.t.\ \mathbb{E}_\pi[g_k]=\delta_k \\ k=1,\dots,K}}{\operatorname{argmin}} KL(\pi, f_i)$$

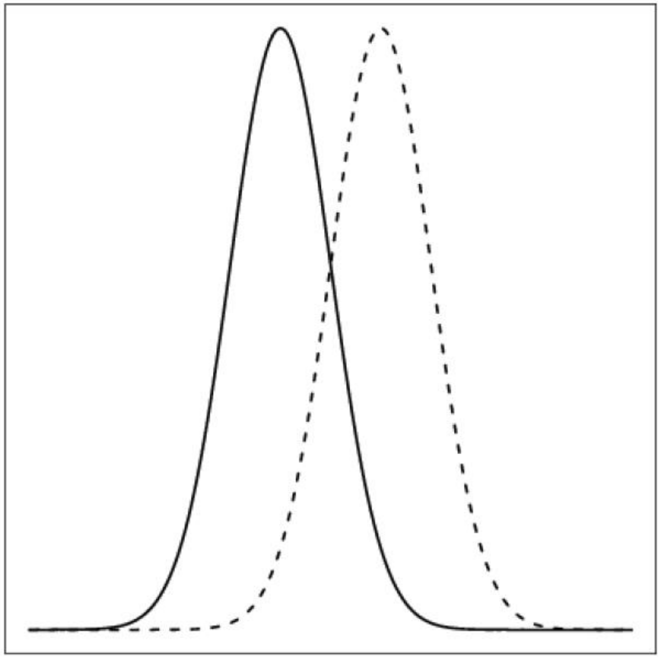
where :

- $g_1, \dots, g_K$ are K linear constraints on the modified density
- and $\delta_1, \dots, \delta_K$ are the values for the perturbed parameters

PERTURBATION EXAMPLES

- **Perturbation of the mean**

$$\mathbb{E}[X_i] = \mu + \delta$$



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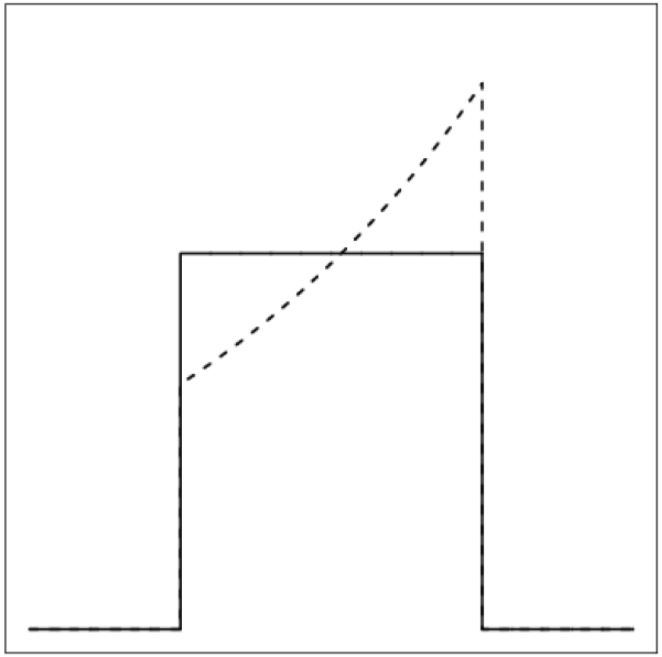
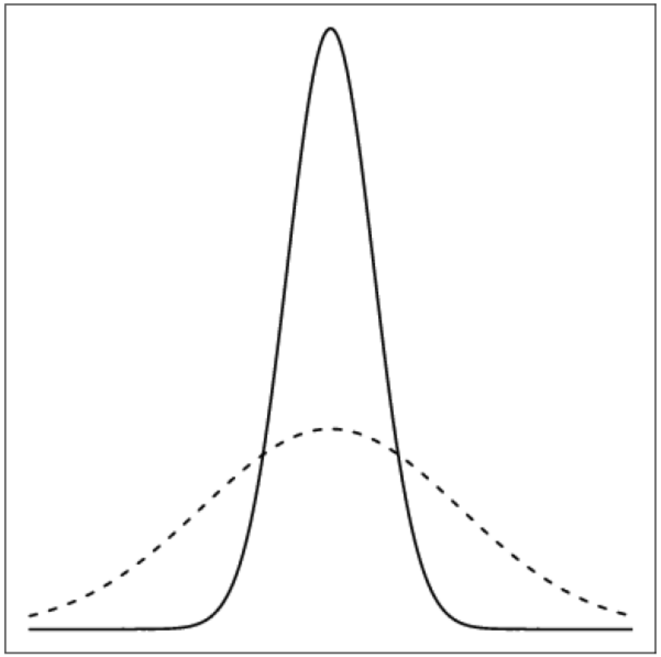


PERTURBATION EXAMPLES

- **Perturbation of the variance**

$$\mathbb{E}[X_i] = \mu$$

$$\mathrm{Var}[X_i] = \mathbb{E}\left[X_i^2\right] - \mu^2 = \sigma^2 + \mu$$



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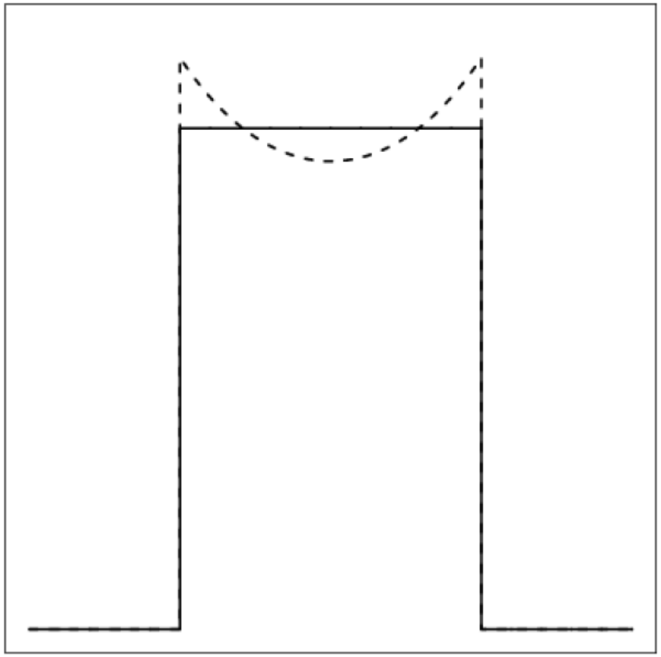
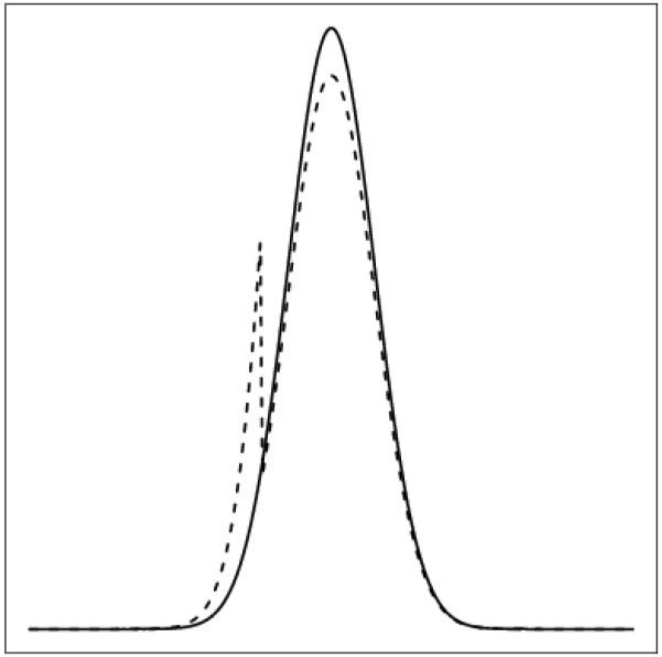


PERTURBATION EXAMPLES

- **Perturbation of a α-quantile**

$$q_\alpha(X_i) = F_i^{-1}(\alpha) = q + \delta$$



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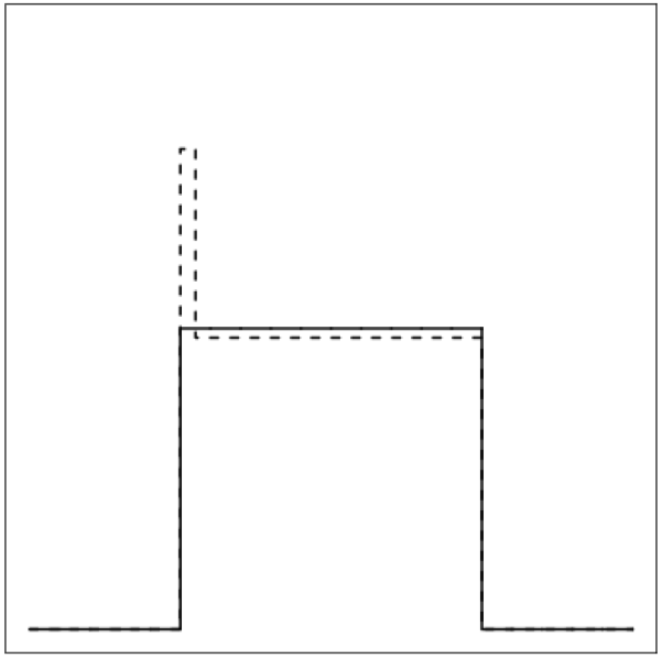

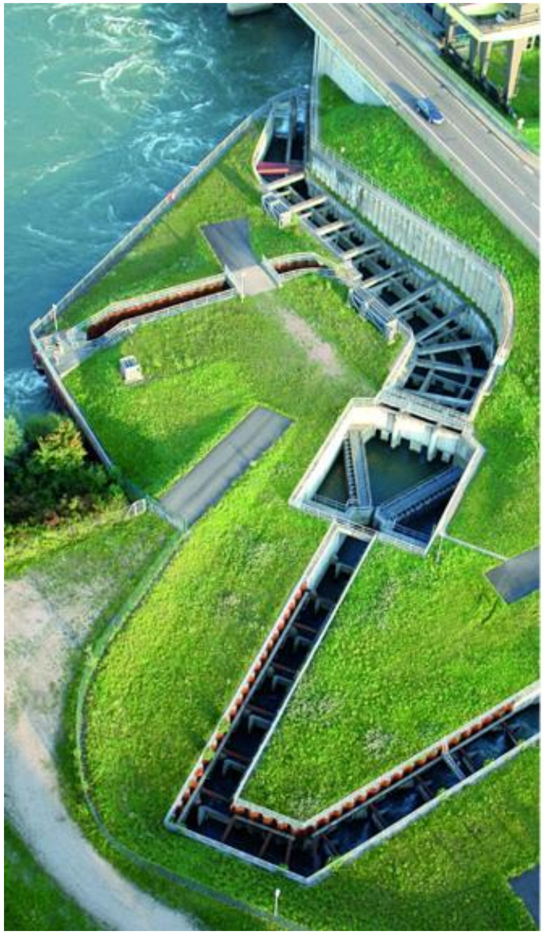

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REVERSE IMPORTANCE SAMPLING

Lemaître et al. (2015)

- **We use a Monte-Carlo framework**

$$\hat{P}_f^N = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}_{\{G(\boldsymbol{x}^{(n)})<0\}}$$

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- **We can estimate $P_{i\delta}$ using the « reverse » importance sampling method**

$$\hat{P}_{i\delta}^N = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}_{\{G(\boldsymbol{x}^{(n)})<0\}} \frac{f_{i\delta}\left(x_i^{(n)}\right)}{f_i\left(x_i^{(n)}\right)}$$

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► $\hat{P}_f^N$ **and** $\hat{P}_{i\delta}^N$ **can be estimated using the same sample**

no need for additional runs of the code G

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no need for additional runs of the code G

- **For a couple $\left(\hat{P}_f^N, \hat{P}_{i\delta}^N\right)$, the plug-in estimator of the $S_{i\delta}$ indice is given by :**

$$\hat{S}_{i\delta}^N = \left(\frac{\hat{P}_{i\delta}^N}{\hat{P}_f^N} - 1\right)\mathbb{I}_{\left\{\hat{P}_{i\delta}^N > \hat{P}_f^N\right\}} + \left(1 - \frac{\hat{P}_f^N}{\hat{P}_{i\delta}^N}\right)\mathbb{I}_{\left\{\hat{P}_{i\delta}^N < \hat{P}_f^N\right\}}$$

ASYMPTOTIC LAW OF THE COUPLE $\left(\hat{P}_f^N, \hat{P}_{i\delta}^N\right)$

Lemaître et al. (2015)

Proposition

if $P_{i\delta} \neq P_f$ and under usual conditions :

i. $\mathrm{Supp}(f_{i\delta}) \subseteq \mathrm{Supp}(f_i)$

ii. $\int_{\mathrm{Supp}(f_{i\delta})} \frac{f_{i\delta}^2(x_i)}{f_i(x_i)} dx_i < +\infty$

$$\sqrt{N}\left(\hat{S}_{i\delta}^N - S_{i\delta}\right) \underset{N \to \infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0, \boldsymbol{\Delta}^T \boldsymbol{\Sigma}_{i\delta} \boldsymbol{\Delta}\right)$$

with :

$$\boldsymbol{\Delta} = \begin{pmatrix} \frac{\partial s}{\partial P_f}\left(P_f, P_{i\delta}\right) \\ \frac{\partial s}{\partial P_{i\delta}}\left(P_f, P_{i\delta}\right) \end{pmatrix} \text{ for } P_{i\delta} \neq P_f \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}}_{i\delta} = \begin{pmatrix} \hat{P}_f^N\left(1 - \hat{P}_f^N\right) & \hat{P}_{i\delta}^N\left(1 - \hat{P}_f^N\right) \\ \hat{P}_{i\delta}^N\left(1 - \hat{P}_f^N\right) & \hat{\sigma}^2{}_{i\delta,N} \end{pmatrix}$$

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- $\hat{S}_{i\delta}^N$ **converges to the true value** $S_{i\delta}$

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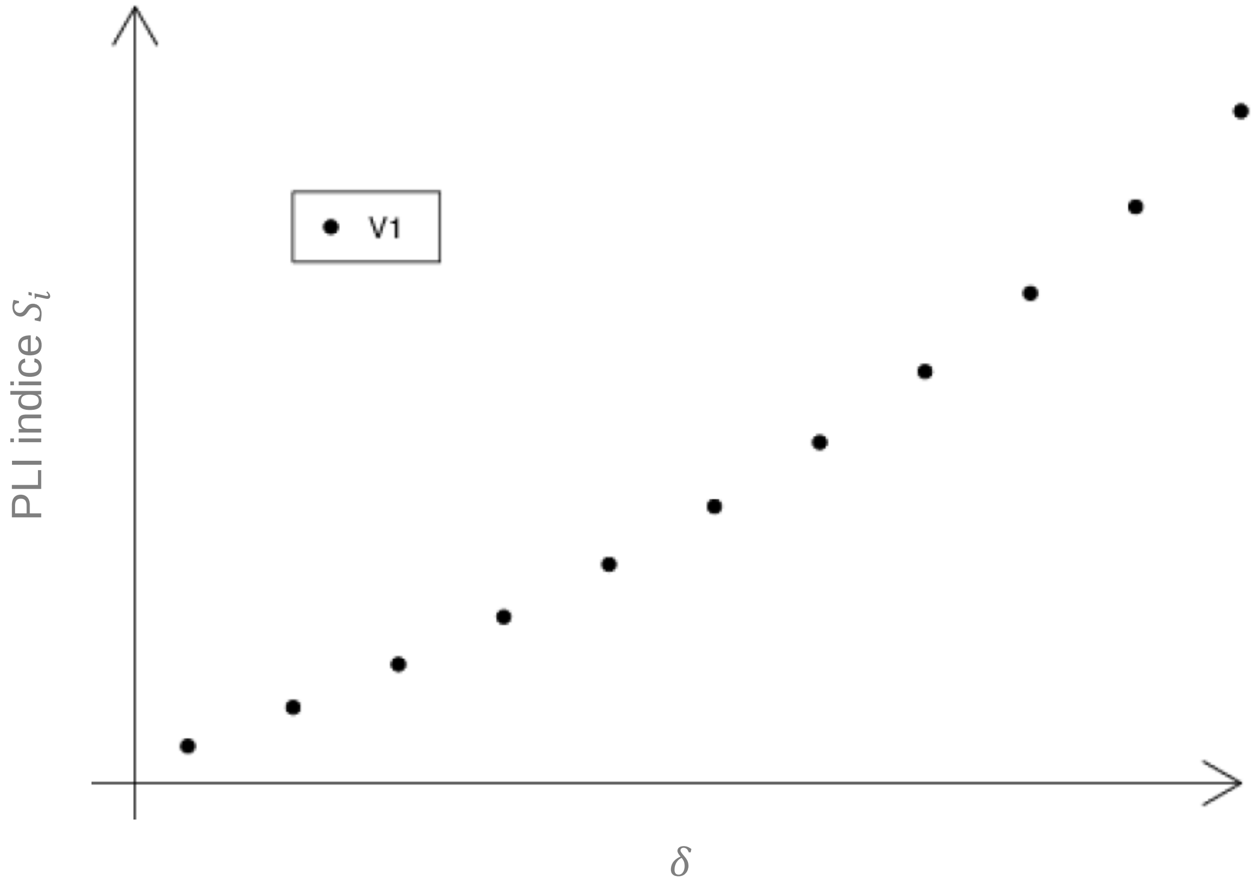
  - **$\hat{S}_{i\delta}^N$ converges to the true value $S_{i\delta}$**
  - **We are able to control the estimation error**

ILLUSTRATION



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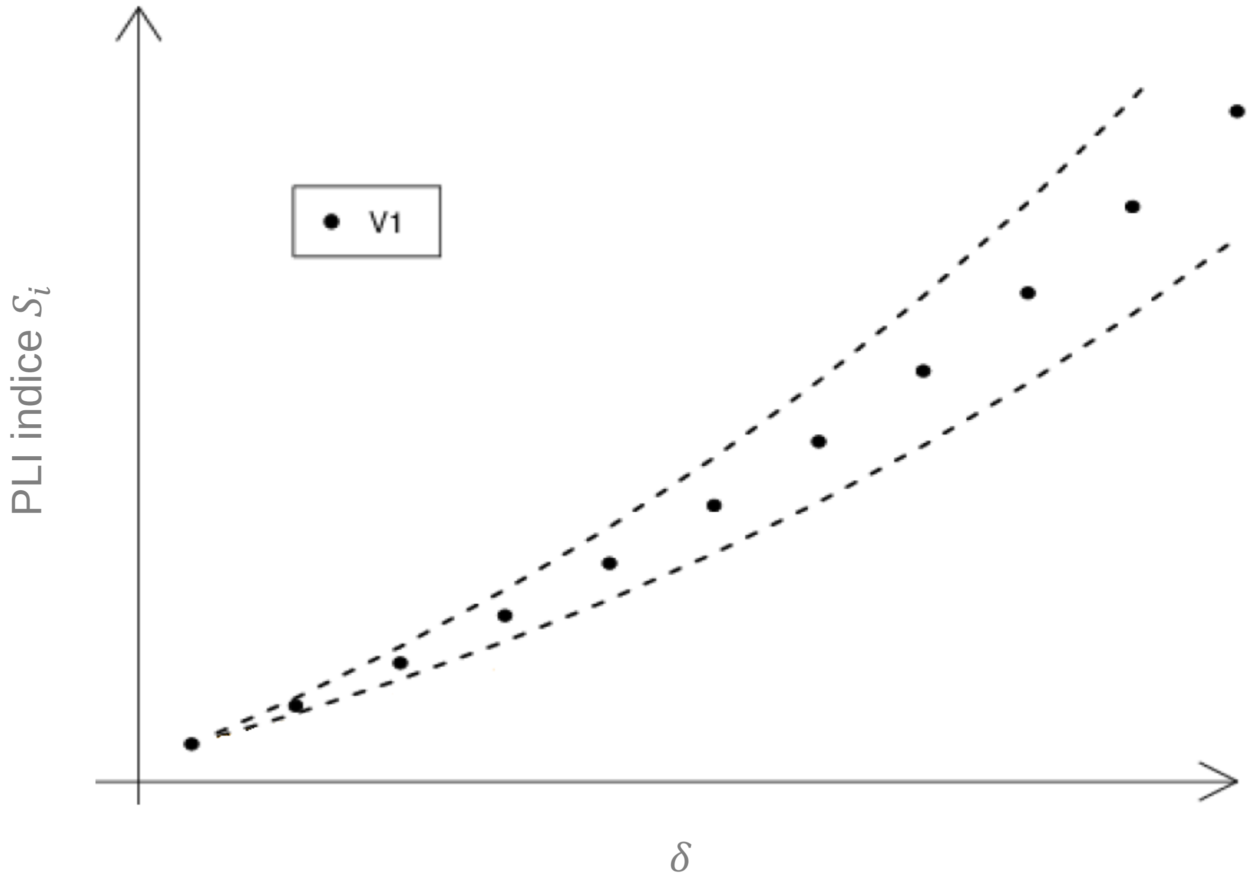

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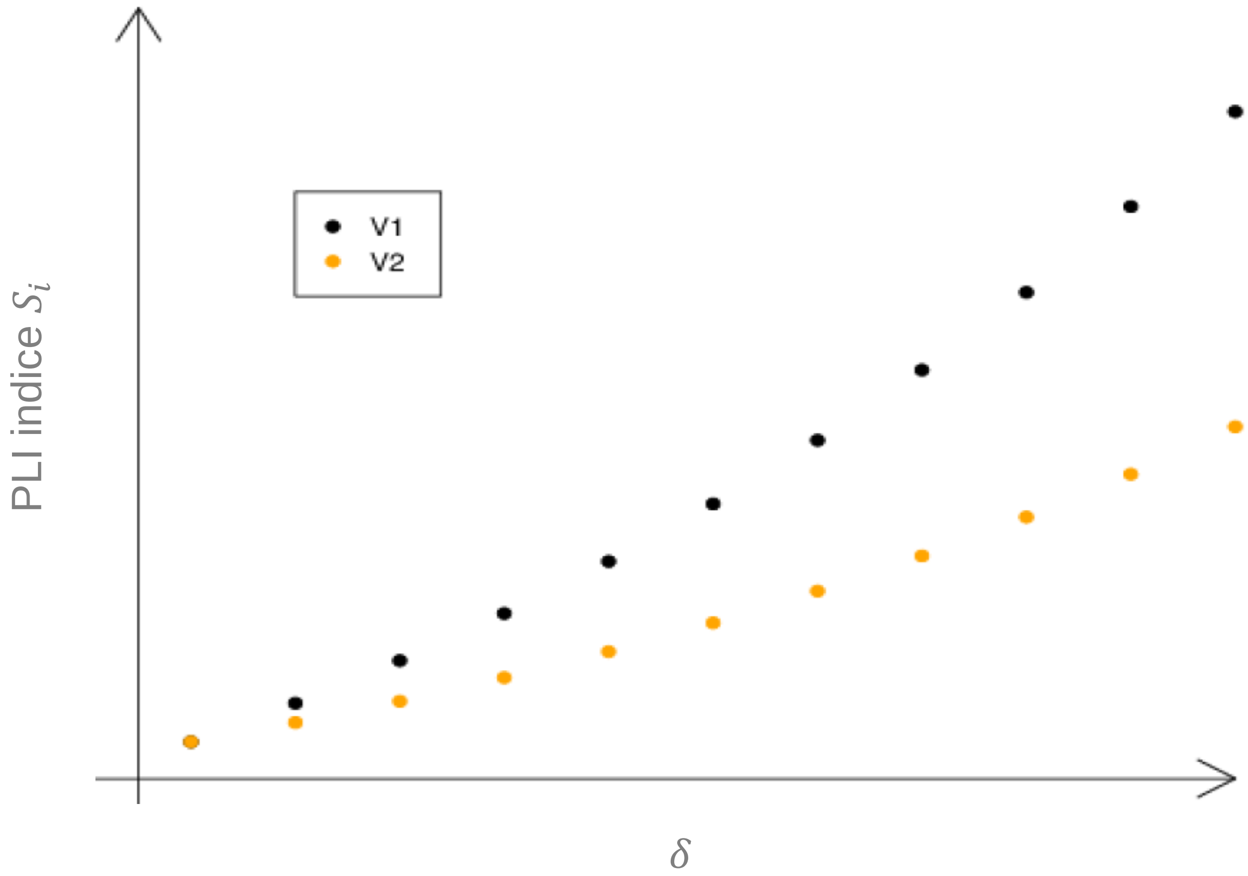

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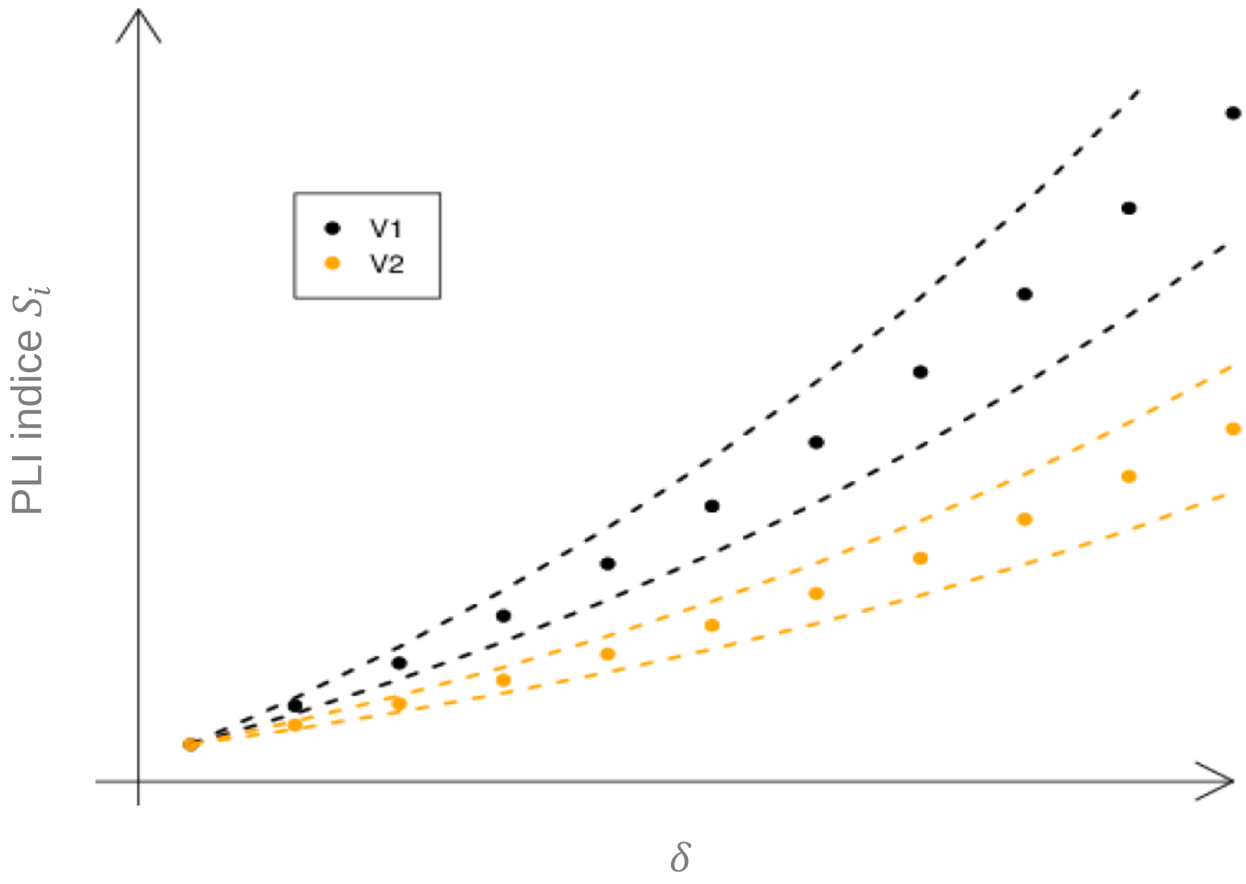

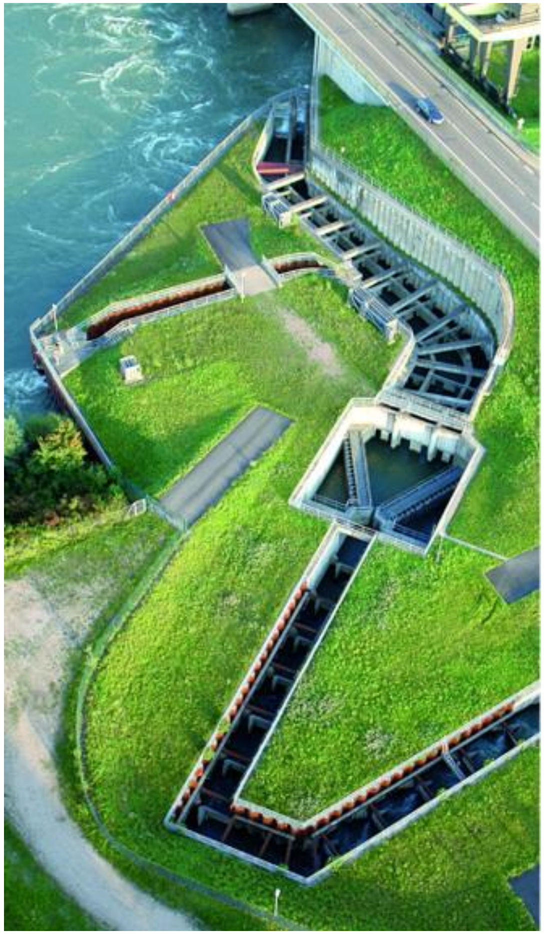

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2. Principle of PLI indices
3. Density perturbation
4. Estimation of perturbed probability
5. **Bayesian approach using GP**

WHY A BAYESIAN APPROACH ?

Difficulties in the estimation of $P_{i\delta}$ and P_f :

- The computer code G is costly (CPU time, memory,…)
- P_f is low ($< 10^{-3}$), sometimes $\approx 10^{-6}$ or 10^{-8}

Hence, we will use :

- an importance sampling method to generate more $\boldsymbol{x}^{(n)}$ in the failure domain $\mathcal{D}_f$ than 1 out of $1/P_f$ (on average)
- a GP meta-model to reduce the number of code runs required for the estimation

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Hence, we will use :

- an importance sampling method to generate more $\boldsymbol{x}^{(n)}$ in the failure domain $\mathcal{D}_f$ than 1 out of $1/P_f$ (on average)
- a GP meta-model to reduce the number of code runs required for the estimation

Two important remarks about IS estimation :

- Desirable property : IS allows unbiased estimation of $P_{i\delta}$ and P_f
- The closer the instrumental density $q(\boldsymbol{x})$ from the optimal one

$$q^*(\boldsymbol{x}) = \frac{\mathbb{I}_{\{G(\boldsymbol{x})<0\}} f(\boldsymbol{x}) d\boldsymbol{x}}{P_f},$$

the more accurate (low variance) the estimation

WHY A BAYESIAN APPROACH ?

Hence, we will use :

- an importance sampling method to generate more $\boldsymbol{x}^{(n)}$ in the failure domain $\mathcal{D}_f$ than 1 out of $1/P_f$ (on average)
- a GP meta-model to reduce the number of code runs required for the estimation

1st possibility : replace G (costly) by a meta-model $\tilde{G}$ in the expression of $\hat{P}_f^N$

$$\hat{P}_f^N = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}_{\{\tilde{G}(\boldsymbol{x}^{(n)})<0\}}$$

► In general, this estimator is biased because of the direct use of $\tilde{G}$

2nd possibility : use $\tilde{G}$ only to define a relevant instrumental density and calculate $\hat{P}_f^N$ using the true G

► allows to take advantage of the regularity of G, without losing the unbiased estimation property

BAYESIAN IMPORTANCE SAMPLING

A two-stages procedure :

1) **Build a surrogate model for G (or $G \times f$) using a $n_0 < n$ budget of code runs and deduce an instrumental density we hope to be close from q^***

2) **Calculate $\hat{P}_f^N$ (and $\hat{P}_{i\delta}^N$) by importance sampling, the $n_1 = n - n_0$ remaining budget, using the instrumental density obtained at the previous stage**

BAYESIAN IMPORTANCE SAMPLING

NIS algorithm : approximation of $G \times f$ by a kernel estimator (Zhang, 1996) or a « Linear Blend Frequency Polygon » (Neddermeyer, 2009)

More recently : procedures using Bayesian approximation for G with gaussian process a priori (Dubourg, 2011), (Auffray *et al.*, 2012), (Dubourg *et al.*, 2013), (Bect & Vazquez, 2013)

BAYESIAN IMPORTANCE SAMPLING

We make the hypothesis the code G is the realization of a GP - Notations are :

$\mathbb{P}_0^G$ a priori law on G

$\mathbb{P}_{n_0}^G$ a posteriori law after the first stage

$\mathbb{P}^q$ law of the $\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(n_1)}$ (iid) generated for the estimation of $\hat{P}_f^N$

$\mathbb{P}_{n_0}^{G,q}$ joint a posteriori law of $G, \boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(n_1)}$

Then, for a cost function $L: \mathbb{R}^2 \rightarrow \mathbb{R}$, we look for :

$$q^*\left(\mathbb{P}_{n_0}^G\right) = \underset{q}{\operatorname{argmin}}\, \mathbb{E}_{n_0}^{G,q}\left[L\left(P_f, \hat{P}_f^{IS,n_1}\right)\right]$$

OPTIMAL DENSITY FOR A QUADRATIC COST FUNCTION

Let's assume that :

$$L : (a, b) \to (a - b)^2$$

and G admits moments of order 2 under $\mathbb{P}_{n_0}^G$

Then the optimal density is :

$$q^*\left(\mathbb{P}_{n_0}^G\right) \propto f \times \sqrt{\mathbb{E}_{n_0}^G[G^2]}$$

We denote $g_{n_0}(\boldsymbol{x}) = \sqrt{\mathbb{E}_{n_0}^G[G(\boldsymbol{x})^2]}$, and $q_{n_0}^* = q^*\left(\mathbb{P}_{n_0}^G\right)$

BAYESIAN ESTIMATOR OF P_f

The constant $Z = \int_{\mathbb{X}} g_{n_0} f$ can be estimated without any call to G

► We estimate it by simple MC of size m (large) : $\hat{Z}_m$

The corresponding estimator of P_f

$$\hat{P}_f^{BIS,n_0,n_1} = \hat{Z}_m \frac{1}{n_1} \sum_{n=1}^{n_1} \frac{G\left(\boldsymbol{X}^{(n)}\right)}{g_{n_0}(\boldsymbol{X}^{(n)}) f(\boldsymbol{X}^{(n)})} f\left(\boldsymbol{X}^{(n)}\right) = \hat{Z}_m \frac{1}{n_1} \sum_{n=1}^{n_1} \frac{G\left(\boldsymbol{X}^{(n)}\right)}{g_{n_0}(\boldsymbol{X}^{(n)})}$$

► Unbiased since $\hat{Z}_m$ is an unbiased estimator of Z, independent on the $\boldsymbol{X}^{(n)}$

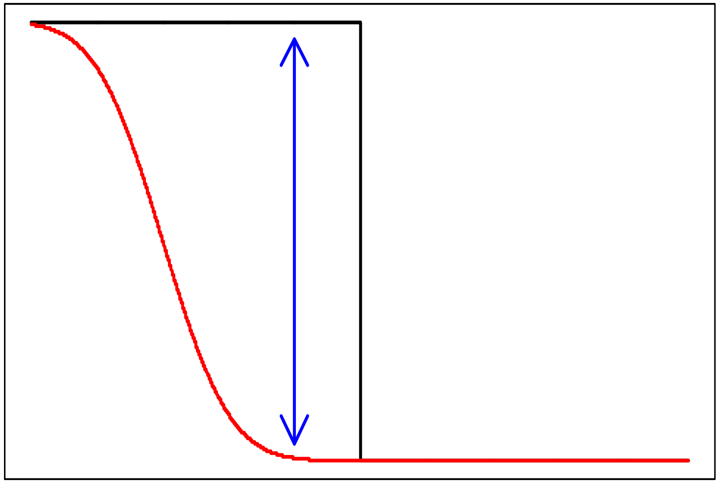
PROBLEM : IF $q^*_{n_0}$ UNDER-ESTIMATES THE OPTIMAL DENSITY $q^*(h)$ ON SOME PARTS OF $\mathbb{X}$

For a perfect model, when $g_{n_0}(\boldsymbol{x}) = G(\boldsymbol{x})$, the theoretical variance is null :

$$\mathrm{Var}\left(\hat{P}_f^{BIS,n_0,n_1}\right) = \frac{1}{n_1}\left(Z\int_{\mathbb{X}} \frac{G^2(\boldsymbol{x})f(\boldsymbol{x})}{g_{n_0}(\boldsymbol{x})}d\boldsymbol{x} - {P_f}^2\right)$$

If g_{n_0} is « biased » the variance tends to $+\infty$

PROBLEM : IF $q^*_{n_0}$ UNDER-ESTIMATES THE OPTIMAL DENSITY $q^*(h)$ ON SOME PARTS OF $\mathbb{X}$



In dimension 1, $\mathbb{I}^2_{\{G(\boldsymbol{x})<0\}}$ in black, $g_{n_0}(\boldsymbol{x})$ in red

SOLUTION : A « DEFENSIVE » SAMPLING

Review of the techniques allowing to control this problem by Owen & Zhou (2000)

Technique of the « defensive mixture » by Hesterberg (1995) :

We add a ε quantity to bound the variance

$$q_\varepsilon(\boldsymbol{x}) = \frac{\left(g_{n_0}(\boldsymbol{x}) + \varepsilon\right) f(\boldsymbol{x})}{Z + \varepsilon}$$

$$= \frac{g_{n_0}(\boldsymbol{x}) f(\boldsymbol{x})}{Z + \varepsilon} + \frac{\varepsilon}{Z + \varepsilon} f(\boldsymbol{x})$$

$$= c_{\boldsymbol{\varepsilon}} f(\boldsymbol{x}) + (1 - c_\varepsilon) q^*(\boldsymbol{x})$$

SOLUTION : A « DEFENSIVE » SAMPLING

The estimator remains unbiased :

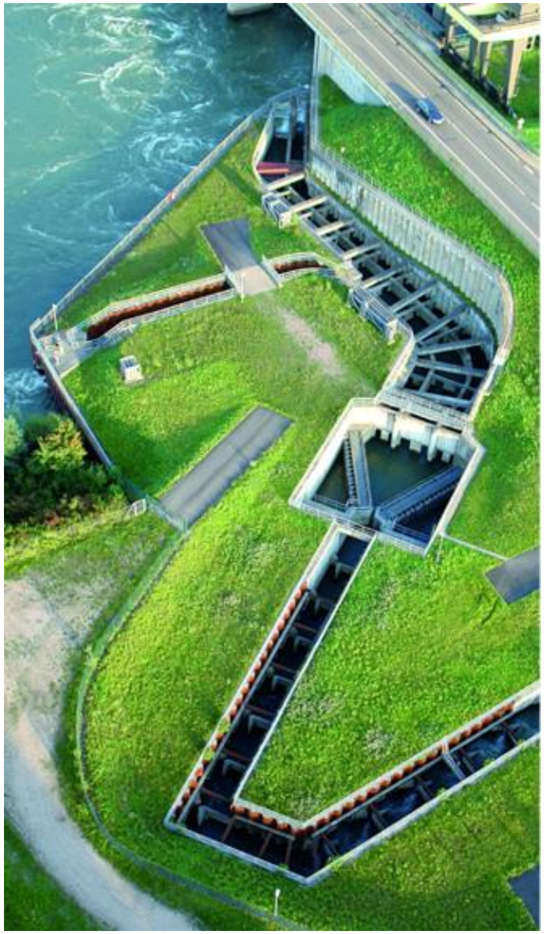
$$\widehat{P}_f^{BIS,n_0,n_1,\varepsilon} = \frac{Z+\varepsilon}{n_1}\sum_{\boldsymbol{n}=1}^{n_1}\frac{G\left(\boldsymbol{X}^{(n)}\right)}{g_{n_0}(\boldsymbol{X}^{(n)})+\varepsilon}, \qquad \boldsymbol{X}^{(n)} \sim q_\varepsilon,$$

with a variance :

$$\mathrm{Var}\left(\widehat{P}_f^{BIS,n_0,n_1,\varepsilon}\right) = \frac{1}{n_1}\left((Z+\varepsilon)\int_{\mathbb{X}}\frac{G^2(\boldsymbol{x})f(\boldsymbol{x})}{g_{n_0}(\boldsymbol{x})+\varepsilon}d\boldsymbol{x} - {P_f}^2\right)$$

$$\leq \frac{1}{n_1}\left(\frac{Z+\varepsilon}{\varepsilon}P_f - {P_f}^2\right)$$

which has a finite value



1. Context : sensitivity analysis on a failure probability
2. Principle of PLI indices
3. Density perturbation
4. Estimation of perturbed probability
5. Bayesian approach using GP
6. **Conclusion**

CONCLUSION

- **We defined a new type of sensitivity indices called PLI adapted to some reliability quantities of interest such as probailities of failure** $P_f = \int_{\mathbb{X}} \mathbb{I}_{\{G(\boldsymbol{x})<0\}} f(\boldsymbol{x}) d\boldsymbol{x}$
- **To this aim we formalized a way of perturbing a law of probability with respect to one of its parameters**
- **We developped an estimation technique of our indices** $S_{i\delta}$ **taking into account our two main constraints :**
  - G is costly
  - P_f is low $\approx 10^{-6}$ or 10^{-8}

CONCLUSION

Perspectives

- **Reach the same kind of theoretical/practical results for PLI indices over quantiles**
- **Take dependences into account**

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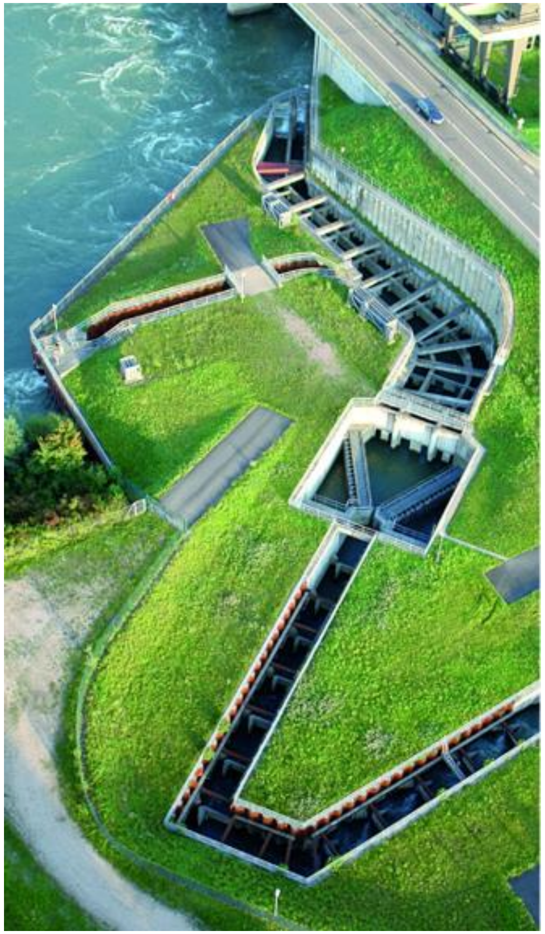
MERCI DE VOTRE ATTENTION

ANNEXES



Expression of the variance of $\widehat{P}_f^{IS,N}$

Demonstration of the optimality of a density $\propto f g_{n_0}$

EXPRESSION OF THE VARIANCE OF $\widehat{P}_f^{IS,N}$

The variance of $\widehat{P}_f^{IS,N}$ is :

$$\mathrm{Var}\left(\widehat{P}_f^{IS,N}\right) = \frac{1}{N}\left(\int_{\mathbb{X}} \frac{G^2 f^2}{q} - {P_f}^2\right) = \frac{{P_f}^2}{N}\left(\int_{\mathbb{X}} \frac{\left(q - q^*(G)\right)^2}{q}\right)$$

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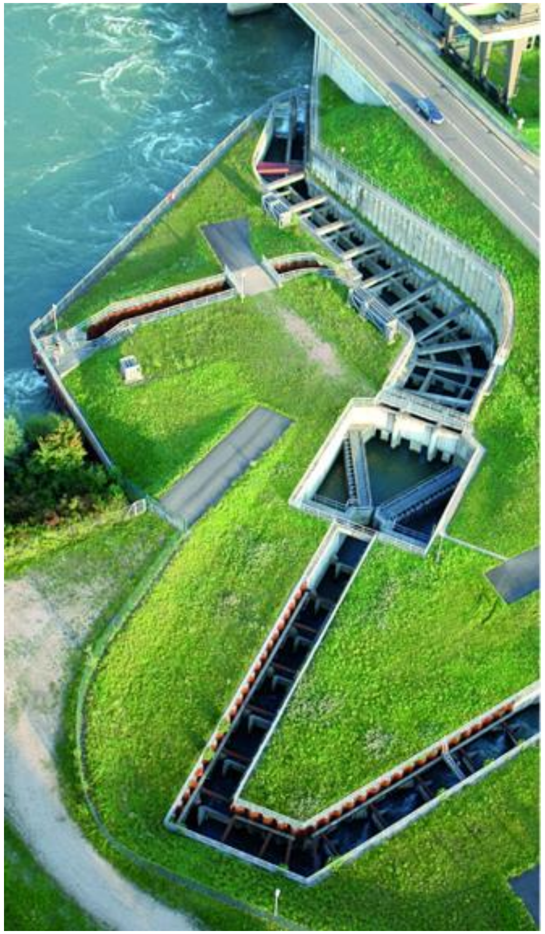
ANNEXES



Expression of the variance of $\widehat{P}_f^{IS,N}$

Demonstration of the optimality of a density $\propto f g_{n_0}$

DEMONSTRATION OF THE OPTIMALITY OF A DENSITY $\propto f g_{n_0}$

We have the following formula for the variance :

$$\mathrm{Var}\left(\widehat{P}_f^{IS,n_1}\right) = \frac{1}{n_1}\left(\int_{\mathbb{X}} \frac{G^2 f^2}{q} - P_f^2\right) = \frac{P_f^2}{n_1}\left(\int_{\mathbb{X}} \frac{\left(q - q^*(G)\right)^2}{q}\right) \qquad (\lozenge)$$

Moreover, by conditionning by the GP G, we get :

$$\mathbb{E}_{n_0}^{G,q}\left[\left(\widehat{P}_f^{IS,n_1} - P_f\right)^2\right] = \mathbb{E}_{n_0}^{G}\left[\mathbb{E}_{n_0}^{q}\left[\left(\widehat{P}_f^{IS,n_1} - P_f\right)^2\right]\right] \qquad (\circ)$$

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DEMONSTRATION OF THE OPTIMALITY OF A DENSITY $\propto f g_{n_0}$

Injecting (◊) into (○), we obtain :

$$\mathbb{E}_{n_0}^{G,q}\left[\left(\widehat{P}_f^{IS,n_1} - P_f\right)^2\right] = \mathbb{E}_{n_0}^{G}\left[\frac{1}{n_1}\left(\int_{\mathbb{X}} \frac{G^2 f^2}{q} - P_f^2\right)\right]$$

$$= \frac{1}{n_1}\left(\int_{\mathbb{X}} \frac{\mathbb{E}_{n_0}^{G}[G^2] f^2}{q} - \mathbb{E}_{n_0}^{G}\left[P_f^2\right]\right)$$

$$= \frac{1}{n_1}\left(\int_{\mathbb{X}} \frac{g_{n_0}^2 f^2}{q} - \mathbb{E}_{n_0}^{G}\left[P_f^2\right]\right) \qquad \textbf{(♣)}$$

DEMONSTRATION OF THE OPTIMALITY OF A DENSITY $\propto f g_{n_0}$

Applying Cauchy-Schwarz inequality to the first term of (♣), appears that :

$$\int_{\mathbb{X}} \frac{g_{n_0}^2 f^2}{q} = \int_{\mathbb{X}} \left(\frac{g_{n_0} f}{q}\right)^2 q \geq \left(\int_{\mathbb{X}} \frac{g_{n_0} f}{q} q\right)^2 = \left(\int_{\mathbb{X}} g_{n_0} f\right)^2 \textbf{(♠)}$$

DEMONSTRATION OF THE OPTIMALITY OF A DENSITY $\propto f g_{n_0}$

Injecting (♠) in (♣), and as this term is the only one that depends on the density q, we show that :

$$\mathbb{E}_{\boldsymbol{n_0}}^{\boldsymbol{G,q}}\left[\left(\widehat{P}_f^{IS,n_1}-\boldsymbol{\alpha}\right)^2\right]=\frac{\mathbf{1}}{\boldsymbol{n_1}}\left(\int_{\mathbb{X}}\frac{\boldsymbol{g_{n_0}^2 f^2}}{\boldsymbol{q}}-\mathbb{E}_{n_0}^{G}\left[P_f^2\right]\right)$$

$$\geq\frac{\mathbf{1}}{\boldsymbol{n_1}}\left(\left(\int_{\mathbb{X}}\boldsymbol{g_{n_0}\pi}\right)^{\mathbf{2}}-\mathbb{E}_{n_0}^{G}\left[P_f^2\right]\right)\qquad (☺)$$

DEMONSTRATION OF THE OPTIMALITY OF A DENSITY $\propto f g_{n_0}$

To conclude, one notices that the equality case is obtained in (♠) and hence in (☺) when q is proportional to $g_{n_0} f$, which means for the instrumental density $q^*\left(\mathbb{P}_{n_0}^G\right)$.