PC Expansion for Global Sensitivity Analysis of non-smooth funtionals of uncertain Stochastic Differential Equation solutions

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Stochastic differential equations (SDEs) play an important role in modeling problems in many different fields. These models are particularly challenging since, in addition to their inherent random dynamics, they are often uncertain due to incomplete knowledge of the parameters and input data, etc. As a result, the model outputs are also uncertain with a variability depending on both the inherent noise and the model uncertainties. The aim of the present work is then to develop efficient techniques to carry out a global sensitivity analysis (GSA) in uncertain SDE driven by Wiener noise. The objective is to quantify the respective contributions, to the total variance of the SDE solution, of the Wiener noise and other sources of parametric uncertainty.

The simplest approach to perform such decomposition of the variance is through a Monte Carlo sampling or any of its variants [2]. Although the implementation of these methods is straightforward, neither MC nor any of its improved versions exploit the potential smoothness with respect to uncertain parameter of the model output, to accelerate the convergence of the sensitivity indices. We propose to use functional approximations (Polynomial Chaos) to account for the dependences on the uncertain model parameters of the random trajectories, as introduced in [1]. Under the assumption that the driving Wiener noise and the uncertain parameters are independent random quantities, the PC expansion can be exploited to perform an orthogonal decomposition of the variance, separating contributions from the uncertain parameters, the Wiener noise, and a coupled contribution. The approach in [1] relied on Galerkin methods to compute the stochastic PC modes of the solution and on the Sobol-Hoeffding decomposition to define the sensitivity indices [5], althought others methods such us Fourier amplitude sensitivity test (FAST) can be applied [3]. In the present work, we propose an extension to non-intrusive or sampling methods for the determination of the PC approximation, along with techniques to perform a GSA of any functional of the SDE solution. In more details, we rely here on a non-intrusive pseudo-spectral projection (PSP) method, over a sparse-grid of parameter points, for the PC modes computation. The GSA of QoIs derived from SDE solution is illustrated in Figure 1. Two different QoIs are considered: the case of path integral which inherit the smoothness of the original SDE solution (see *Left*) and the case of exit time which exhibit non smooth dependences with the uncertain parameters (see *Right*).

For the first case, the non-intrusive projection of the QoI can be directly performed, and the sensitivity indices can be computed from the random PC modes. Numerical experiments show that the standard error (SE) of the sensitivity indices for the direct approach can be much less than for the SE of the classical MC estimation [4] of the sensitivity indices, when the same number of Wiener noise samples are used. This improvement in the SE comes at the cost of having to solve as many SDEs as sparse grid points in the parameter space; so the reduction of the variance in the sensitivity indices estimators may not necessarily translates into an improvement of the overall computational efficiency, in particular in the case of complex parametric dependencies demanding large sparse grids.

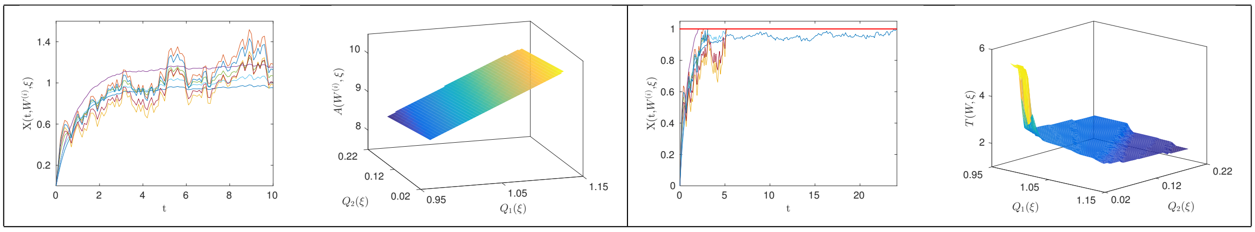

Figure 1: Ornstein-Uhlenbeck (OU) process with uncertain parameters: $dX(W, \boldsymbol{\xi}) = (Q_1(\boldsymbol{\xi}) - X(W, \boldsymbol{\xi}))dt + (\nu X(W, \boldsymbol{\xi}) + 1)Q_2(\boldsymbol{\xi})dW$, $X(t = 0) = 0$, with parameters $Q_1 \sim \mathscr{U}([0.95, 1.15])$, $Q_2 \sim \mathscr{U}([0.02, 0.22])$ and $\nu = 0.2$. *Left:* Integral over time of X (smooth QoI). *Right:* Exit time of X at level 1(non-smooth QoI).

For the second case, the QoI (exit time) does not inherit the smoothness of the SDE solution with respect to the uncertain parameters, and its direct non-intrusive projection has a very slow convergence. Therefore, we introduce the idea of *indirect* non-intrusive projection which consists in, first, constructing the approximation of the SDE solution and, second, sampling this approximation to generate conditional samples of the QoI. This approach yields a lower standard error than the classical MC estimator when the same number of noise samples is used, as can be seen in Figure 2. However, the cost of the improvement might again be significant for some problems. Our numerical experiments show that for problems with a low noise effects and low number of sparse grid points, the indirect projection is more efficient that direct MC, providing lower SE for the same cost. In addition, for the indirect projection gives access to a richer information than direct MC sampling (for instance all sensitivity indices can be computed without requiring more noise samples).

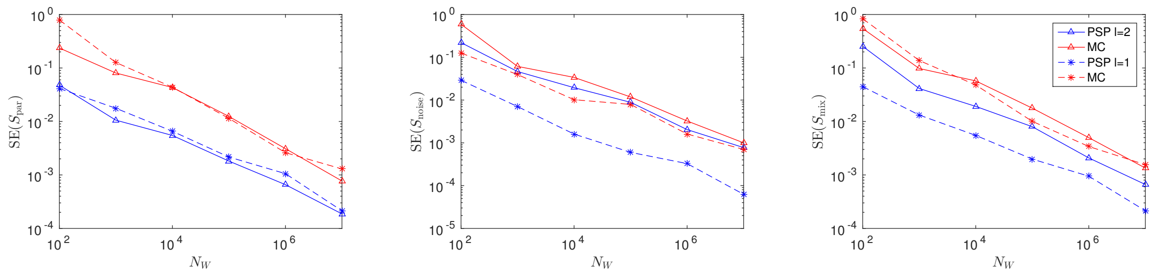

Figure 2: Standard errors in the sensitivity indices of the exit time, as a function of the number of samples N_W, for the indirect PSP and standard MC approaches. The solid lines correspond to the multiplicative noise OU process with large noise, while the dashed lines correspond to the low additive noise case.

References

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