# VARIANCE-BASED SENSITIVITY INDICES FOR INPUTS DEFINED OVER NON-RECTANGULAR DOMAINS.

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Most of the literature on variance-based methods for sensitivity analysis focuses on independent inputs. In recent years, interest has increased for correlated input. However, in both cases the domain of the inputs is rectangular, such as a hypercube of size one. In practical cases, it happens that the domain of the inputs is not rectangular but has a different shape, such as a circular area, or a triangle.

Let us denote by $x = (y, z)$ a vector of inputs of dimension *d* where y and z are two sub-sets of inputs of size *s*<*d* and *d-s*. $x$isassumed distributed with probability density function $p(y, z)$over the unit hypercube$I^d(0,1)$. The marginal distribution of *y* is indicated with $p(y)$ and the conditional distribution of y given z is indicated with $p(y|z)$. The output is assumed scalar and is given by $f(y, z)$, meaning that it is obtained by evaluating the function *f* at point *x*.

First order $S_y$ and total order $S_y^T$sensitivity indices for input *y* are defined in [1]:

$$S_y = \frac{1}{D}\left[\int_{R^n} f(y',z')p(y',z')dy'dz'\left[\int_{R^{n-s}} f(y',\hat{z})p(y',\hat{z}\,|\,y')d\hat{z} - \int_{R^n} f(y,z)p(y,z)dydz\right]\right]$$

$$S_y^T = \frac{1}{2D}\int_{R^{n+s}}[f(y,z) - f(\bar{y}',z)]^2 p(y,z)p(\bar{y}',z\,|\,z)dyd\bar{y}'dz$$

where$(y, z)$ and $(y', z')$ are two independent random vectors generated from the joint distribution $p(y, z)$, $\hat{z}$ is a random vector generated from the conditional probability density function $p(y', \hat{z}|y')$ and $(\bar{y}', z)$ is a random vector generated from the conditional probability density function $p(\bar{y}', z|z)$.

In the present study we propose four approaches to compute variance-based sensitivity indices in the case of non-rectangular domains and compare their performance. As benchmark, we consider the *g*-function in two settings and we analyse two types of non-rectangular domain: circular, at different radii, and triangular in different conditions. The analytic sensitivity indices for all case studies have been computed using the formulas above and have been used as the reference for the benchmark.

- The first approach uses the classic Sobol' formula for the uncorrelated case as proposed in [1] coupled with a rejection method that considers the points only if they are inside the domain. In this case, we test to what extent this formula is valid as the domain gets less and less rectangular.
- The second approach uses the Sobol' formula for correlated inputs proposed by [2] coupled with a rejection method proposed in [3].
- The third and fourth approaches evaluate the sensitivity indices using the definition, i.e. computing the variance of the conditional expectation of the model output, given one of the inputs, and dividing by the total output variance. Both approaches start from a set of Monte Carlo points and divide the range of a given input in bins.

While in the third method the same number of points in each bin is considered (implying that the bins can have different size), the fourth method uses the same bin size.

References

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[2] Kucherenko S, S. Tarantola and P. Annoni(2012)Estimation of global sensitivity indices for models with dependent variables, Computer Physics Communications, Vol: 183, Pages: 937-946

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