Dynamic sensitivity analysis method based on Gramian

S. HAMZA
Université de Lorraine, CRAN, Nancy, France

F. ANSTETT-COLLIN
Université de Lorraine, CRAN, Nancy, France

A. BIROUCHE
Université de Haute-Alsace, MIPS, Mulhouse, France

Context

A dynamic vehicle depends on various subsystems which characterize the vehicle behavior. Each subsystem is described by a mathematical model depending on a significant number of parameters. These parameters are very often uncertain due to a lack of measurements, knowledge due to expert judgment. The uncertainty in the parameters propagates through the model and manifests itself at the model output. In order to understand the vehicle behavior, it is essential to know the parameters responsible for the model output variation. Uncertainty and sensitivity analysis can help to evaluate the impact of this lack of knowledge on the model response ([1,2]). In the literature, sensitivity analysis for dynamical models is not straightforward. In this context, the work presented in this paper investigates a novel technique of global sensitivity analysis for dynamical models. The originality of the method is to use control theory tools for sensitivity analysis purposes.

Methodology

Consider a dynamical linear system presented in state space form given by:

$$\sum_{SYS} : \left\{ \begin{array}{l} \dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) \\ y(t) = C(\theta)x(t) + D(\theta)u(t) \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $y(t) \in \mathbb{R}^{n_p}$ the output vector, $u(t) \in \mathbb{R}^{n_m}$ the input vector and $t \in \mathbb{R}^+$ refers to the time. $A(.) \in \mathbb{R}^{n_x \times n_x}$ is the state matrix, $B(.) \in \mathbb{R}^{n_x \times n_p}$ the input matrix, $C(.) \in \mathbb{R}^{l \times n_x}$ the output matrix and $D(.) \in \mathbb{R}^{l \times n_p}$ the feedforward matrix. The vector $\theta = [\theta_1, \ldots, \theta_{n_\theta}]$ represents the n_θ uncertain parameters. As the parameters θ_i are uncertain, they are considered as random variables defined by their probability density function (uniform,Gaussian, etc.). The uncertainty of the parameters is propagated through the model on the output $y(t)$ which becomes also uncertain. The aim is to determine the most influential parameters θ_i on the output uncertainty.

The proposed method is based on the analysis of the system energy required to drive a state to a final one by the input. If this energy is minimal, the system is said controllable. Controllability means that the system dynamics can be modified when acting on the input signal $u(t)$. The system energy depends on the uncertain parameters θ. Intuitively, if the parameter variation leads to a significant variation of the energy, it means that this parameter variation acts on the system dynamics and leads to a system that is more or less controllable. In this case, the system controllability is sensitive to this parameter variation and thus this parameter is influential on the system states. The system energy can be determined through the reachability Gramian ([3]).

The infinite time reachability Gramian, denoted for short $W_R(\theta)$, when $t_f \to \infty$ is given by:

$$W_R(\theta) = \lim_{t \to \infty} W_R(\theta, t_0, t_f) = \int_0^\infty e^{A(\theta)t} B(\theta) B(\theta)^T e^{A(\theta)^T t} dt \tag{2}$$

and it is obtained by solving the continuous-time Lyapunov equation :

$$A(\theta)W_R(\theta) + W_R(\theta)A(\theta)^T + B(\theta)B(\theta)^T = 0 \tag{3}$$

The minimal energy allowing to bring a system to a final state x_f, since $x(t_0) = 0$, is given by:

$$\|u\|_2 = \int_0^\infty u^T(t)u(t)dt = x_f^T W_R(\theta)^{-1} x_f \quad (4)$$

The quantity $x_f^T W_R^{-1} x_f$ in (4) represents an hyperellipsoid which includes all the reachable states obtained from the optimal input sequence u_{opt}. This quantity depends on the inverse of the reachability Gramian $W_R^{-1}(\theta)$. Each eigenvalue of $W_R^{-1}(\theta)$, denoted λ_i, corresponds to one system state. In fact, these eigenvalues determine the size of the axes of the hyperellipsoid and the eigenvectors determine its directions. Intuitively, the variation of an influential parameter will lead to a significant change of the dimension of the hyperellipsoid axes (see FIGURE 1).

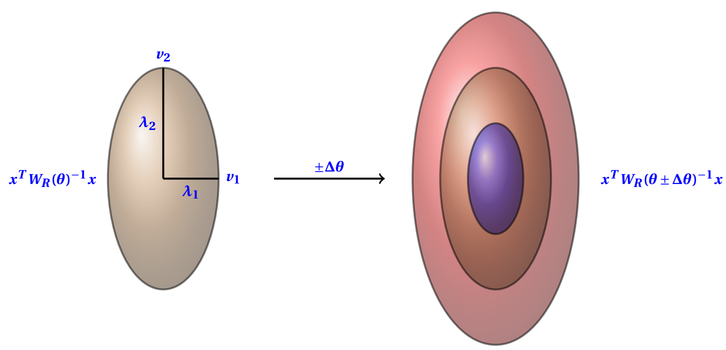

FIGURE 1 – Example of second-order system energy variation according to parameters variation.

According to (2) and (4), the eigenvalues of W_R^{-1} provide information on how controllable the system is. Higher the eigenvalues are, lower the required energy is and thus more controllable the system is ([3]). If any $\lambda_i = 0$, the system is not controllable. In fact, each eigenvalue λ_i is a function of the system parameters θ. The eigenvalues represent a measure of the controllability, that is the sensitivity of the system dynamics to variation. In this way, the parameters involved in the expression of the eigenvalues are influential on the system states. If the eigenvalue does not depend on a given parameter, this parameter is not influential on the state variation. From the structural expression of $W_R^{-1}(\theta)$, the qualitative influence of the parameters can be deduced. Then, to quantify the individual contribution of each parameter to the variance of the energy $\|u\|_2$, the Sobol' indices can be computed.
The advantage of the proposed approach is that the system energy does not depend on time and is thus scalar. Furthermore, it takes into account the global model behavior.
The approach is applied on a bicycle model describing dynamic behavior of the vehicle.

References

[1] A. Saltelli and K. Chan and E. M. Scott, "Sensitivity analysis",Wiley, 2000.

[2] E. De Rocquigny and N. Devictor and S. Tarantola, "Uncertainty in industrial practice", Wiley, 2008.

[3] B. Moore, Principal component analysis in linear systems : Controllability, observability, and model reduction. Automatic Control, IEEE Transactions on (1981) 17–32.